Viscous-Gravity Spreading of Oil on Water: Modeling and Challenges

R. Chebbi

***Abstract*—Oil spreading is one of the major factors affecting the fate of oil spills on water. Modeling spreading is required to study the impact of oil slicks on the environment and plants using sea water including desalination units. Spreading of oil on water undergoes three stages. In the second stage, gravity acts as the main driving force against the viscous force, which is the main resisting force in stages 2 and 3. The paper presents the state of the art in modeling the second stage of spreading. Challenges in analyzing viscous-gravity spreading of continuously discharged oil on water are also presented.**

***Index Terms*—Oil, slick, spill, spreading, viscous gravity, water.**

I. Introduction

The fate of oil slicks spilled on the sea is affected by several factors including spreading, advection, evaporation, dissolution, dispersion, water-in-oil emulsification, photolysis, sinking and biodegradation [1], [2]. The order of magnitude analysis [3]-[5] shows the existence of three stages of oil spreading on water: the inertia-gravity regime, followed by the viscous-gravity phase, and finally the viscous-surface tension stage, where in each phase there is one predominant resisting effect to one predominant promoting force. The second phase is considered in the present paper. Previous experimental work was performed for the unidirectional constant volume case in [5] and for axisymmetric and unidirectional steady oil discharge cases in [6], [7]. The theoretical work in [5] considers the constant oil volume case and assumes a velocity profile in the oil phase changing in the direction of spreading but with no variation across the oil layer depth (vertical direction), along with a boundary-layer approach for the flow in the water phase assuming a boundary layer thickness that is dependent upon time but uniform along the direction of spreading. This last assumption was relaxed in [8], [9]. The theoretical treatment of the variable oil discharge case was considered in [10]. Approximate treatments for the laminar boundary layer were used in [11] based on the well-known Blasius solution to solve for the dynamics of spreading in both cases of constant and variable oil-spill volumes. The present paper reviews the different findings and provides directions for future work based on the present challenges.

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II. Order-of-Magnitude Analysis

A schematic of an oil spill spreading on water is shown in Fig. 1.

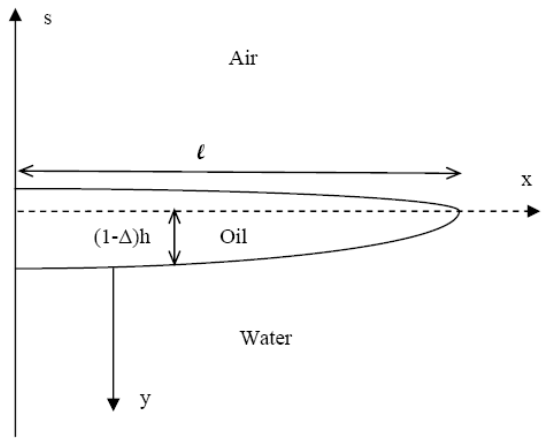

Fig. 1. Schematic of the cross section of half of a symmetric oil spill.

The analysis in [3]-[5] shows the existence of three stages of spreading: the inertia-gravity, viscous-gravity and viscous-surface tension stages. The analysis is based on the order of magnitudes for the two resisting (inertia and viscous) forces, and the two promoting (gravity and surface-tension) forces. Two spreading cases are considered: unidirectional (ζ=0) and axisymmetric (ζ=1). The volume V (taken as half of the volume per unit length in the case ζ=0) is either constant (m=0) or equal to:

$$V = q\,t \quad (m=1) \tag{1}$$

In case the oil-spill volume increases continuously at a constant rate q (variable volume case, m=1). The size is defined as the radius in the axisymmetric case and as half of the width in the unidirectional spreading case.

For the spreading stage considered, gravity forces are balanced by viscous forces with the orders of magnitude given by:

$$\text{Viscous force } \left[\mu_w \left(\ell t^{-1}\right)\left(\nu_w t\right)^{-1/2}\right] \ell^{\zeta+1} \tag{2}$$

$$\text{Gravity force } \left(\rho_w\, \Delta\, g\, h_c\right) h_c\, \ell^{\zeta} \tag{3}$$

where t is time, μ_w and ν_w are the dynamic and kinematic viscosities, respectively, and is a characteristic oil thickness, satisfying the relationship

$$h_c\, \ell^{\zeta} \sim V \tag{4}$$

The spreading laws are summarized below, along with the order of magnitudes of the transition time and spill size from the first stage of spreading to the second one.

$$\ell \sim \left(\frac{\Delta\, g\, V^2}{\nu_w^{1/2}}\, t^{3/2} \right)^{1/4} ;\; T = \left(\frac{V^4}{g^2 \Delta^2\, \nu_w^3} \right)^{1/7}$$

$$L = \left(\frac{g \Delta V^5}{\nu_w^2} \right)^{1/7} \quad (m=0,\ \zeta=0) \tag{5}$$

$$\ell \sim \left(\frac{\Delta\, g\, V^2}{\nu_w^{1/2}}\, t^{3/2} \right)^{1/6} ;\; T = \left(\frac{V}{g \Delta\, \nu_w} \right)^{1/3}$$

$$L = \left(\frac{g \Delta V^5}{\nu_w^2} \right)^{1/12} \quad (m=0,\ \zeta=1) \tag{6}$$

$$\ell \sim L \left(\frac{t}{T} \right)^{7/8} ;\; T = \frac{1}{\nu_w} \left(\frac{q^4}{g^2 \Delta^2} \right)^{1/3} ;\; L = \frac{1}{\nu_w} \left(\frac{q^5}{g \Delta} \right)^{1/3}$$

$$(m=1,\ \zeta=0) \tag{7}$$

$$\ell \sim L \left(\frac{t}{T} \right)^{7/12} ;\; T = \left(\frac{q}{g \Delta \nu_w} \right)^{1/2} ;\; L = \left(\frac{q^5}{g \Delta \nu_w^3} \right)^{1/8}$$

$$(m=1,\ \zeta=1) \tag{8}$$

The prefactor η_m (proportaionality constant) in each of the above equations needs further analysis as it is undetermined by the order-of-magnitude analysis.

III. Governing Equations

The pressure distribution in the vertical direction is hydrostatic, leading to the fraction floating above the mean water level [5], [9]

$$\Delta = \frac{\rho_w - \rho_o}{\rho_w} \tag{9}$$

which is the relative density difference between the water density ρ_w and the oil density ρ_o.

A. Change in Velocity in the Oil Phase

Integration of the oil momentum balance, along with order-of-magnitude analysis provides the following results [9], [10].

$$\Delta U/U \sim \frac{\mu_w}{\mu_o} \left(\frac{t}{T} \right)^{-7/8} \quad (m=0,\ \zeta=0) \tag{10}$$

$$\Delta U/U \sim \frac{\mu_w}{\mu_o} \left(\frac{t}{T} \right)^{-1} \quad (m=0,\ \zeta=1) \tag{11}$$

$$\Delta U/U \sim \frac{\mu_w}{\mu_o} \left(\frac{t}{T} \right)^{-3/8} \quad (m=1,\ \zeta=0) \tag{12}$$

$$\Delta U/U \sim \frac{\mu_w}{\mu_o} \left(\frac{t}{T} \right)^{-2/3} \quad (m=1,\ \zeta=1) \tag{13}$$

where μ_o denotes the oil viscosity. Typically oil viscosity is significantly larger than the water viscosity; therefore, changes in velocity in the oil phase, ΔU, can be neglected compared to the interfacial velocity U.

B. Oil Momentum and Continuity Equations

The change in velocity in the oil phase being small, the continuity equation leads to:

$$\frac{\partial h}{\partial t} = -\frac{1}{x^\zeta} \frac{\partial \left(x^\zeta\, U\, h \right)}{\partial x} \tag{14}$$

The oil thickness profile is bound to satisfy the following condition

$$\int_0^\ell (2\pi x)^\zeta\, h\, dx = V \tag{15}$$

The lubrication theory approximation is used in the oil phase, and the vertical component of the oil is considered negligible compared to the component in the direction of spreading. The oil momentum equation reduces to [5], [9]

$$0 = \mu_o \frac{\partial^2 U}{\partial s^2} - \frac{\partial p}{\partial x} \tag{16}$$

Integration of the oil momentum equation, along with the continuity of the tangential stress condition at the water-oil interface and the zero shear condition at the oil-air interface leads to [5], [9]

$$-\mu_w \left. \frac{\partial u}{\partial y} \right|_{y=0} + \rho_w \Delta\, g\, h \frac{\partial h}{\partial x} = 0 \tag{17}$$

C. Water Momentum and Continuity Equations

A boundary-layer model is adopted [5] with the x-momentum equation and continuity equation given by

$$\frac{\partial u}{\partial t} + u\, \frac{\partial u}{\partial x} + v\, \frac{\partial u}{\partial y} = \nu_w\, \frac{\partial^2 u}{\partial y^2} \tag{18}$$

$$\frac{1}{x^\zeta} \frac{\partial \left(x^\zeta\, u \right)}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{19}$$

where u and v are the horizontal and vertical components of the velocity in the water phase (see Fig. 1). The no-slip condition forces u and U to be equal at the water-oil interface

$$u = U \quad \text{at } y = 0 \tag{20}$$

IV. Integral Momentum Equation

Applying the von-Kármán integral technique leads to [9]

$$\frac{\partial}{\partial \bar{t}} \left(\int_0^\delta \bar{u}\, d\bar{y} \right) + \frac{\partial}{\partial \bar{x}} \left(\int_0^\delta \bar{u}^2\, d\bar{y} \right) + \frac{\zeta}{\bar{x}} \int_0^\delta \bar{u}^2\, d\bar{y} = - \left. \frac{\partial \bar{u}}{\partial \bar{y}} \right|_{\bar{y}=0} \tag{21}$$

where the dimensionless variables are defined as

$$\bar{t} = t/T\ ;\ \bar{x} = x/L\ ;\ \bar{y} = y/\sqrt{\nu_w T}\ ;\ \bar{h} = h/\left(V/L^{\zeta+1}\right);$$
$$\bar{u} = u/(L/T)\,;\bar{v} = v/\sqrt{\nu_w/T}\ ;\ \bar{U} = U/(L/T) \tag{22}$$

A sixth-order polynomial profile is selected to approximate the water velocity variation along the vertical direction

$$\frac{u}{U} = \frac{\bar{u}}{\bar{U}} = \sum_{i=0}^{6} a_i \left(\frac{\bar{y}}{\delta}\right)^i \tag{23}$$

In (23) δ is the boundary-layer thickness divided by $\sqrt{\nu_w T}$. The polynomial coefficients are determined using the following boundary conditions [9]

$$\bar{u} = \bar{U}\ ; \quad \frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u}\ \frac{\partial \bar{u}}{\partial \bar{x}} = \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$

$$\frac{\partial}{\partial \bar{y}}\left(\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u}\ \frac{\partial \bar{u}}{\partial \bar{x}} + v\ \frac{\partial \bar{u}}{\partial \bar{y}}\right) = \frac{\partial^3 \bar{u}}{\partial \bar{y}^3} \text{ at } \bar{y} = 0 \tag{24}$$

$$\bar{u} = \frac{\partial \bar{u}}{\partial \bar{y}} = \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} = \frac{\partial^3 \bar{u}}{\partial \bar{y}^3} = 0 \quad \text{at } \bar{y} = \delta \tag{25}$$

V. Similarity Solution

A. Similarity Variables

A similarity solution is sought for oil thickness, oil velocity and boundary-layer thickness profiles in [9], [10] as defined by

$$\bar{U} = \frac{\bar{x}}{\bar{t}}\, C(\eta) \tag{26}$$

$$\tilde{\delta}(\eta) = \frac{\delta}{\sqrt{\bar{t}}} \tag{27}$$

$$\bar{G} = \left[\frac{\bar{h}\ \bar{t}^{\,n(\zeta+1)}}{\eta_m}\right]^2 \tag{28}$$

where n is the power-law exponent in the spreading laws in (5)-(8), η_m is the spreading-law prefactor, and the combined variable η is defined as

$$\eta = \frac{\bar{x}}{\bar{t}^{\,n}} \tag{29}$$

For convenience, the following variables are defined [9]-[10]

$$\tilde{\delta}_s = \tilde{\delta}^2, \bar{\eta} = \frac{x}{\ell} = \frac{\eta}{\eta_m} \tag{30}$$

B. Oil Velocity Profile

The value of C is constant in the case of constant oil volume, given by [5], [9], [10] providing a linear oil velocity profile.

$$C = n \quad (m=0) \tag{31}$$

In the case of a steady oil discharge (m=1), C is not constant, with a variation given by [10]

$$\frac{dC}{d\bar{\eta}} = (n - C)\left(\frac{1}{2\,\bar{G}}\frac{d\bar{G}}{d\bar{\eta}} + \frac{\zeta+1}{\bar{\eta}}\right) - \frac{1}{\bar{\eta}} \qquad (m=1) \tag{32}$$

C. Boundary-Layer Thickness Profile

The rate of change is given by [10]

$$\frac{d\tilde{\delta}_s}{d\bar{\eta}} = \frac{N}{D} \tag{33}$$

where the numerator N and denominator D are given by

$$N = \frac{-a_1\ C}{\bar{\eta}} + \left(\frac{C\ \alpha}{2} - 2\ C^2\ \beta - \zeta\ C^2\ \beta\right)\frac{\tilde{\delta}_s}{\bar{\eta}} - \tilde{\delta}_s\,\frac{dC}{d\bar{\eta}}\left(-n\,\alpha + 2\,\beta\ C\right) \tag{34}$$

$$D = \left(-\frac{n}{2}\ C\ \alpha + \frac{C^2\ \beta}{2} - n\ C\ \tilde{\delta}_s\,\frac{d\alpha}{d\tilde{\delta}_s} + C^2\ \tilde{\delta}_s\,\frac{d\beta}{d\tilde{\delta}_s}\right) \tag{35}$$

And α and β are rational functions of $\tilde{\delta}_s$ [9]. The C-derivative term in (34) is zero in the constant oil volume case.

D. Oil Thickness Profile

The rate of change is given by [9]-[10]

$$\frac{d\bar{G}}{d\bar{\eta}} = 2\,\bar{\eta}\ C\,\frac{a_1}{\tilde{\delta}} \tag{36}$$

Integration in the constant oil volume case leads to

$$\bar{G} = 2\,C\int_1^{\bar{\eta}} (a_1\bar{\eta}/\tilde{\delta})\,d\bar{\eta} \quad (m=0) \tag{37}$$

E. Spreading-Laws Prefactors

The relationship between the oil volume and the oil thickness profile gives [9]-[10]

$$\eta_m = \left[(2\,\pi)^{\zeta}\int_0^1 \sqrt{\bar{G}}\ \bar{\eta}^{\zeta}\ d\bar{\eta}\right]^{-1/(\zeta+2)} \tag{38}$$

VI. VI. Results and Discussions

The solution is performed using Runge-Kutta integration to solve for $\bar{G}$, C (m=1 case only) and $\tilde{\delta}_s$ [9], [10].

A. Constant Oil Volume

The case m=0 is easier to solve. Using $\bar{G} = 0$, and $\tilde{\delta}_s = 0$ at the leading edge ($\bar{\eta} = 1$), integration is performed till the origin is reached. The results show a constant boundary layer thickness and a nearly constant oil thickness with significant variations near the leading edge for both profiles. The results obtained in [9] are compared with the experimental data and the theoretical work in [8] (using finite differences) in Table I, showing that the constant-volume case can be considered as well understood at least at the laboratory scale.

TABLE I: COMPARISON BETWEEN EXPERIMENTAL AND THEORETICAL RESULTS FOR H_M (M=0)

Spreading case	Theoretical [8]	Theoretical [9]	Experimental [4]-[5]
Unidirectional	1.76	1.77	1.5

The axisymmetric case results show consistency between the detailed boundary-layer treatment of the laminar boundary- layer flow in the water phase (η_m=1.15) in [9] and the approximate analysis in [11] based on Blasius solution (η_m=1.09).

B. Variable Oil Volume

The existence of a point where both N and D in Eq. (33) become very small makes it necessary to perform two integrations, one starting from the leading edge and the other from the origin, and to "match" the two solutions [10]. The existence of singularities at both ends requires analytical solutions to initiate integration near both ends of the integration domain [10].

1) Unidirectional case

The solution leads to a nearly constant velocity profile throughout the whole oil phase and a quite constant thickness profile with significant variations near the leading edge. The spreading law prefactors results are summarized in Table II. The theoretical values are consistent, and deviate significantly from the experimental value. This is discussed in the next section.

TABLE II: COMPARISON BETWEEN EXPERIMENTAL AND THEORETICAL RESULTS FOR H_M (M=1)

Spreading case	Theoretical [10]	Theoretical [11]	Experimental [7]
Unidirectional	1.19	1.09	0.572

2) Axisymmetric case

Matching was not possible between the solutions originating from both ends of the integration domain (leading edge and origin). On the other hand, the theoretical work in [11] shows significant deviation from the experimental value in [6] (see Table III). Both aspects are discussed in the next section.

TABLE III: COMPARISON BETWEEN EXPERIMENTAL AND THEORETICAL RESULTS FOR H_M (M=1)

Spreading case	Theoretical [11]	Experimental [6]
Axisymmetric	0.816	0.363

VII. CHALLENGES (VARIABLE VOLUME CASE) AND CONCLUSIONS

The failure to match the two solutions in the axisymmetric case is attributed to the fact that the asymptotic solution leads to infinite velocity at the origin which gives large boundary layer thickness for the solution originating from the origin, and therefore no possible matching with the solution originating from the leading edge [10].

The requirement that surface tension force is small compared to gravity imposes the conditions [10]

$$\frac{\sigma}{\rho_w (g\Delta)^{1/3} q^{4/3}} << 1 \quad (\zeta = 0)$$

$$\frac{\sigma}{\rho_w (g\Delta q \nu_w)^{1/2}} << 1 \quad (\zeta = 1) \tag{39}$$

which are not satisfied in the two sets of experiments [6], [7].

This clearly shows the need for additional experimental that could use the experimental design and technique in [6]-[7] or modified ones, while ensuring that surface tension will not play any significant role by satisfying the above-mentioned requirements. From a theoretical point of view, the full treatment of the boundary-layer flow field in the axisymmetric variable-volume case is still open due to the above mentioned reason.

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